# CONSTRUCTION OF PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH SEVERAL DEGREES OF FREEDOM FOR PARTICULAR CASES

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The case in which the equation for the frequencies has simple roots has been considered in [1,2]. The case of multiple roots has been considered for systems with two degrees of freedom in [3]. The present work deals with the cases when the equation for the frequencies has multiple and zero roots.

**1. Case of multiple positive roots of the equation of frequencies.** Let us consider a vibrating system with $n$ degrees of freedom for which the equation of motion has the form

$$\sum_{k=1}^{n} (a_{ik}x_k^{\cdot\cdot} + c_{ik}x_k) = \mu F_i (x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, \mu) \qquad (1.1)$$
$$a_{ik} = a_{ki}, \qquad c_{ik} = c_{ki} \qquad (i = 1, \ldots, n)$$

Here and in what follows, we use the notation $dx/dt = x^{\cdot}$.

The functions $F_i$ are assumed to be analytic in their arguments within some region, and the parameter $\mu$ is small. The generating system ($\mu = 0$) is a linear conservative system with constant coefficients. The kinetic and potential energies of the system are assumed to be expressible in terms of positive definite quadratic forms. Under these conditions all roots of the frequency equation

$$\Delta (\omega^2) = | c_{ik} - \omega^2 a_{ik} | = 0 \qquad (1.2)$$

are positive. Suppose that among these roots there is one root of multiplicity $l$, for example, let $\omega_1^2 = \omega_2^2 = \ldots \omega_l^2$, while the remaining $n - l$ roots are simple ones.

The particular solutions of the generating system for simple or multiple positive roots $\omega_r^2$ of the equation of the frequencies have, as is well known, the form

$$x_{k0}^{(r)}(t) = A_{kr}\cos\omega_r t + B_{kr}\,\omega_r^{-1}\sin\omega_r t \tag{1.3}$$

The coefficients $A_{kr}$ (or $B_{kr}$) are determined by the system of linear equations

$$\sum_{k=1}^{n}(c_{ik} - \omega_r^2 a_{ik})A_{kr} = 0 \qquad (i = 1, \ldots, n;\ r = 1, \ldots, n) \tag{1.4}$$

For the root $\omega^2 = \omega_1^2$ of multiplicity $l$, not only is the determinant $\Delta(\omega^2)$ equal to zero but so are all of its derivatives with respect to $\omega^2$ up to and including the $(l - 1)$-st one. Since the determinant $\Delta(\omega^2)$ is symmetric, all of its minors of orders $n - l + 1$ and less are also equal to zero. Hence, for the root $\omega_1^2$ only $n - l$ equations of system (1.4) are independent, the remaining $l$ equations are implied by these $n - l$ equations. Hence for this root there remain $l$ quantities $A_{kr}$ (analogously for $B_{kr}$) undetermined, i.e. arbitrary.

Let us arrange the equations in system (1.4) so that for the multiple root $\omega_1^2$ the first $l$ equation are consequences of the remaining $n - l$ equations. The solution of the generating system will have the following structure:

$$x_{k0}(t) = \sum_{r=1}^{l} q_k^{(r)}x_0^{(r)}(t) + \sum_{r=l+1}^{n} p_k^{(r)}x_0^{(r)}(t) \qquad (k = 1, \ldots, n) \tag{1.5}$$

For $k = 1, \ldots, l$, we have

$$q_k^{(r)} = 1 \qquad (r = k), \qquad q_k^{(r)} = 0 \qquad (r \neq k) \tag{1.6}$$

The remaining coefficients $q_k^{(r)}$ with $k = l + 1, \ldots, n$, can be found by solving the system of the last $n - l$ equations (1.4), with $i = l + 1, \ldots, n$, for $A_{l+1}, \ldots, A_{n1}$ when $\omega = \omega_1$. These quantities will be linear functions of $A_{11}, \ldots, A_{l1}$ with coefficients equal to $q_k^{(r)}$

$$A_{k1} = q_k^{(1)}A_{11} + \ldots + q_k^{(l)}A_{l1} \qquad (k = l + 1, \ldots, n) \tag{1.7}$$

The quantities $p_k^{(r)}$ are determined by the formulas

$$p_k^{(r)} = \frac{\Delta_{ik}(\omega_r^2)}{\Delta_{i1}(\omega_r^2)} \qquad (k = 1, \ldots, n;\ r = l + 1, \ldots, n) \tag{1.8}$$

The functions $x_0^{(r)}(t)$ which enter into formulas (1.5) have the form

$$x_0^{(r)}(t) = A_r \cos \omega_r t + B_r \omega_r^{-1} \sin \omega_r t \qquad (r = 1, \ldots, n) \tag{1.9}$$

Let us assume that the frequency $\omega_l$ is commensurate with the frequencies $\omega_{l+1}, \ldots, \omega_h$, and that it is not commensurate with any of the other frequencies. It is obvious that to the frequencies $\omega$, $\omega_{l+1}, \ldots, \omega_h$ there correspond some solutions, of the generating system, with period $T_0$. We construct the periodic solution of system (1.1) which for $\mu = 0$ goes over into the mentioned periodic solution of the generating system. Hereby the resulting periodic solution will have the period $T = T_0 + \alpha$, where $\alpha$ is a function of $\mu$ which vanishes when $\mu = 0$.

By the method of a small parameter the initial conditions for system (1.1) are obtained from the initial conditions of the generating system by adding to them linear combinations of certain functions $\beta_r(\mu)$ and $\gamma_r(\mu)$ which vanish when $\mu = 0$. One can show, in a manner analogous to the one used in [1] for the case of simple roots of the equation of frequencies, that these functions $\beta_r(\mu)$ and $\gamma_r(\mu)$ can be introduced in such a way that they appear everywhere as the sums $A_r + \beta_r$ and $B_r + \gamma_r$. Therefore the initial conditions for system (1.1) may be taken in the following form [2]:

$$\begin{aligned} x_k(0) &= \sum_{r=1}^{l} q_k^{(r)}(A_r + \beta_r) + \sum_{r=l+1}^{h} p_k^{(r)}(A_r + \beta_r) + \sum_{r=h+1}^{n} p_k^{(r)}\varphi_{r-h} \\ x_k^{\cdot}(0) &= \sum_{r=2}^{l} q_k^{(r)}(B_r + \gamma_r) + \sum_{r=l+1}^{h} p_k^{(r)}(B_r + \gamma_r) + \sum_{r=h+1}^{n} p_k^{(r)}\psi_{r-h} \end{aligned} \tag{1.10}$$

Since system (1.1) is autonomous, one of the pairs $B_r + \gamma_r$ which enter into initial conditions (1.10) can be set equal to zero. In the given case it is assumed that $B_1 = 0$ and $\gamma_1 = 0$. The quantities $\varphi_{r-h}$ and $\psi_{r-h}$ will be analytic functions of $A_p + \beta_p$ and $B_q + \gamma_q$, and $\mu$

$$\begin{aligned} \varphi_{r-h} &= \varphi_{r-h}(A_p + \beta_p, B_q + \gamma_q, \mu) \\ \psi_{r-h} &= \psi_{r-h}(A_p + \beta_p, B_q + \gamma_q, \mu) \end{aligned} \qquad \begin{pmatrix} p = 1, \ldots, h \\ q = 2, \ldots, h \end{pmatrix} \tag{1.11}$$

where

$$\varphi_{r-h}(A_p + \beta_p, B_q + \gamma_q, 0) = 0, \qquad \psi_{r-h}(A_p + \beta_p, B_q + \gamma_q, 0) = 0$$

The solution of system (1.1) can be represented in the form [2]

$$x_k(t, A_r + \beta_r, B_r + \gamma_r, \mu) = q_k^{(1)}(A_1 + \beta_1)\cos \omega_1 t +$$

$$+ \sum_{r=2}^{l} q_k^{(r)} \left[ (A_r + \beta_r) \cos \omega_1 t + \frac{B_r + \gamma_r}{\omega_1} \sin \omega_1 t \right] +$$

$$+\sum_{r=l+1}^{h} p_k^{(r)}\left[(A_r+\beta_r)\cos\omega_r t+\frac{B_r+\gamma_r}{\omega_r}\sin\omega_r t\right]+$$
$$+\sum_{r=h+1}^{n} p_k^{(r)}\left(\varphi_{r-h}\cos\omega_r t+\frac{\psi_{r-h}}{\omega_r}\sin\omega_r t\right)+$$
$$+\sum_{m=1}^{\infty}\left[C_{km}(t)+\sum_{r=1}^{h}\frac{\partial C_{km}}{\partial A_r}\beta_r+\sum_{r=2}^{h}\frac{\partial C_{km}}{\partial B_r}\gamma_r+\cdots\right]\mu^m \qquad (1.12)$$

The functions $C_{km}(t)$ satisfy [1] the system of equations

$$\sum_{k=1}^{n}(a_{ik}\ddot{C}_{km}+c_{ik}C_{km})=H_{im}(t),\qquad H_{im}(t)=\frac{1}{(m-1)!}\left(\frac{d^{m-1}F_i}{d\mu^{m-1}}\right)_{\beta_r=\gamma_r=\mu=0} \qquad (1.13)$$
$$(i=1,\ldots,n)$$

under the vanishing initial conditions: $C_{km}(0) = \dot{C}_{km}(0) = 0$. Solving this system of equations by the operational method, we obtain

$$C_{km}(t)=\frac{1}{\Delta^*(D^2)}\sum_{i=1}^{n}\Delta_{ik}^*(D^2)\,H_{im}(t)\qquad\left(D=\frac{d}{dt}\right) \qquad (1.14)$$

Here we have the obvious relations

$$\Delta^*(D^2)=\Delta(-\omega^2),\qquad \Delta_{ik}^*(D^2)=\Delta_{ik}(-\omega^2)$$

Next we have

$$\Delta^*(D^2)=\Delta_0(D^2+\omega_1^2)^l\prod_{r=l+1}^{n}(D^2+\omega_r^2)=(D^2+\omega_1^2)^l\Delta^{\circ *}(D^2)$$
$$\Delta_0=|a_{ik}|$$

Because of the properties of a symmetric determinant

$$\Delta_{ik}^*(D^2)=(D^2+\omega_1^2)^{l-1}\Delta_{ik}^{\circ *}(D^2)$$

Thus we obtain

$$\frac{\Delta_{ik}^*(D^2)}{\Delta^*(D^2)}=\frac{\Delta_{ik}^{\circ *}(D^2)}{(D^2+\omega_1^2)\Delta^{\circ *}(D^2)}=\frac{1}{\Delta_0}\left[\frac{L_{ik}^{(1)}}{D^2+\omega_1^2}+\sum_{r=l+1}^{n}\frac{L_{ik}^{(r)}}{D^2+\omega_r^2}\right] \qquad (1.15)$$

It is easy to see that for $r = l + 1, \ldots, n$

$$L_{ik}^{(r)}=\Delta_{ik}(\omega_r^2)\left[\prod_{s\neq r}^{n}(\omega_s^2-\omega_r^2)\right]^{-1}=K_{ik}^{(r)}$$

Here the $K_{ik}^{(r)}$ are coefficients which have been obtained earlier for simple roots of the equation of frequencies [1]. For the multiple roots $\omega_1^2$ we obtain

$$L_{ik}^{(1)} = \overset{\circ}{\Delta}_{ik}(\omega_1^2)\,[\prod_{s=l+1}^{n}(\omega_s^2 - \omega_1^2)]^{-1}$$

In this manner we derive

$$C_{km}(t) = C_{km}^{(1)}(t) + \sum_{r=l+1}^{n} p_k^{(r)} C_m^{(r)}(t) \tag{1.16}$$

The functions $C_m^{(r)}(t)$, with $r = l + 1, \ldots, n$, are determined by means of formulas (14) and (15) of [1]. Let us consider the function $C_{km}^{(1)}(t)$ which corresponds to the multiple root $\omega_1^2$

$$C_{km}^{(1)}(t) = [\Delta_0\omega_1 \prod_{s=l+1}^{n}(\omega_s^2 - \omega_1^2)]^{-1}\int_0^t R_{km}^{(1)}(\tau)\sin\omega_1(t-\tau)\,d\tau$$

$$R_{km}^{(1)}(t) = \sum_{i=1}^{n}\overset{\circ}{\Delta}_{ik}(\omega_1^2)\,H_{im}(t)$$

From the condition

$$\frac{A_{k1}}{A_{11}} = \frac{\Delta_{ik}(\omega_1^2)}{\Delta_{i1}(\omega_1^2)} = \frac{\overset{\circ}{\Delta}_{ik}(\omega_1^2)}{\overset{\circ}{\Delta}_{i1}(\omega_1^2)} \qquad (k = 2, \ldots, n)$$

we deduce

$$\frac{A_{k1}}{\overset{\circ}{\Delta}_{ik}(\omega_1^2)} = \ldots = \frac{A_{11}}{\overset{\circ}{\Delta}_{i1}(\omega_1^2)}$$

Hence, taking into account formulas (1.6) and (1.7) we obtain

$$\overset{\circ}{\Delta}_{ik}(\omega_1^2) = q_k^{(1)}\overset{\circ}{\Delta}_{i1}(\omega_1^2) + \ldots + q_k^{(l)}\overset{\circ}{\Delta}_{il}(\omega_1^2)$$

Thus we find

$$C_{km}^{(1)}(t) = \sum_{r=1}^{l} q_k^{(r)} C_m^{(r)}(t) \qquad (k = 1, \ldots, n) \tag{1,17}$$

where

$$C_m^{(r)}(t) = [\Delta_0\omega_1 \prod_{s=l+1}^{n}(\omega_s^2 - \omega_1^2)]^{-1}\int_0^t R_m^{(r)}(\tau)\sin\omega_1(t-\tau)\,d\tau$$

$$R_m^{(r)}(t) = \sum_{i=1}^{n}\overset{\circ}{\Delta}_{ir}(\omega_1^2)\,H_{im}(t) \qquad (r = 1, \ldots, l) \tag{1.18}$$

Hence we obtain

$$C_{km}(t) = \sum_{r=1}^{l} q_k^{(r)} C_m^{(r)}(t) + \sum_{r=l+1}^{n} p_k^{(r)} C_m^{(r)}(t) \tag{1.19}$$

The coefficients $q_k^{(r)}$ with $k = 1, \ldots, l$ are determined with the aid of formulas (1.6) while for $k = l + 1, \ldots, n$ they are derived in accordance with the arguments given on the first page of this work.

Thus the solution of system (1.1) for the case of one multiple root of the equations of frequencies can be presented in a form which is analogous to the form of the solution of a linear generating system

$$x_k(t) = \sum_{r=1}^{l} q_k^{(r)} x^{(r)}(t) + \sum_{r=l+1}^{n} p_k^{(r)} x^{(r)}(t) \qquad (k = 1, \ldots, n) \tag{1.20}$$

Since the frequency $\omega_1$ is commensurate only with the frequencies $\omega_{l+1}, \ldots, \omega_h$, the function $x^{(r)}(t)$ has the form

$$x^{(r)}(t) = (A_r + \beta_r)\cos\omega_r t + \frac{B_r + \gamma_r}{\omega_r}\sin\omega_r t + X^{(r)}(t)$$

$$B_1 = 0, \qquad \gamma_1 = 0 \qquad (r = 1, \ldots, h) \tag{1.21}$$

$$x^{(r)}(t) = \varphi_{r-h}\cos\omega_r t + \frac{\psi_{r-h}}{\omega_r}\sin\omega_r t + X^{(r)}(t) \quad (r = h + 1, \ldots, n)$$

where

$$X^{(r)}(t) = \sum_{m=1}^{\infty}\left[C_m^{(r)}(t) + \sum_{j=1}^{h}\frac{\partial C_m^{(r)}}{\partial A_j}\beta_j + \sum_{j=2}^{h}\frac{\partial C_m^{(r)}}{\partial B_j}\gamma_j + \ldots\right]\mu^m \qquad (r = 1, \ldots, n) \tag{1.22}$$

On the basis of [2], the stated problem on the construction of a periodic solution of period $T_0 + \alpha$ for system (1.1) under the auxiliary conditions formulated above, reduces to the successive solving of two problems: (a) the construction of a periodic solution for a system with $h$ degrees of freedom, (b) the computation of additional corrections for this solution. The method of computation of these correcting terms is shown in [2] on an example involving a system with two degrees of freedom.

Up to now we have assumed that the equation of frequencies had just one multiple root. The extension of the procedure used to the case of several multiple roots presents no difficulties.

**2. Case of zero roots of the equation of frequencies.** Let us assume that the equation of frequencies (1.2) has, in addition to positive roots, a zero root which may be either a simple or a multiple root. A particular solution corresponding to this root can be obtained from expression (1.3) if one lets the frequency go to zero in this expression. Then

$$x_{k0}^{(r)} = A_{kr} + B_{kr}t \tag{2.1}$$

The structure of the solution of the generating system is not changed by the presence of zero roots. It is obvious that the structure of the solution of system (1.1) is not changed either.

The method of constructing periodic solutions of system (1.1) remains the same; however, the computational formulas in the present case will differ somewhat from the earlier ones. In particular, a term of a new type appears when the quantity $\Delta_{ik}^*(D^2)/\Delta^*(D^2)$, which occurs in the computation of the coefficients $C_{km}(t)$, is broken up into partial fractions. For example, if there is one zero root $\omega_n = 0$, then

$$\frac{\Delta_{ik}^*(D^2)}{\Delta^*(D^2)} = \frac{1}{\Delta_0}\left[\sum_{s=1}^{n-1} \frac{K_{ik}^{(s)}}{D^2 + \omega_s^2} + \frac{K_{ik}^{(n)}}{D^2}\right]$$

where

$$K_{ik}^{(n)} = \Delta_{ik}(0)\left[\prod_{s=1}^{n} \omega_s^2\right]^{-1}$$

Hereby the last term which is contained in the function $C_{km}(t)$ will have the form

$$C_{km}(t) = \ldots + \Delta_{ik}(0)\left[\Delta_0 \prod_{s=1}^{n-1} \omega_s^2\right]^{-1} \int_0^t R_{km}^{(n)}(\tau)(t-\tau)\, d\tau \qquad (2.2)$$

The characteristic peculiarities of the construction of a periodic solution in the presence of a zero root of the equation of frequencies will be revealed by an example of a system with two degrees of freedom. We consider the system

$$\begin{aligned} a_{11}x_1^{\cdot\cdot} + a_{12}x_2^{\cdot\cdot} + c_{11}x_1 + c_{12}x_2 &= \mu F_1(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}, \mu) \\ a_{21}x_1^{\cdot\cdot} + a_{22}x_2^{\cdot\cdot} + c_{21}x_1 + c_{22}x_2 &= \mu F_2(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}, \mu) \end{aligned} \qquad (2.3)$$

Let the equation of frequencies of the generating system have the roots $\omega_1^2$ and 0. In what follows we shall omit the subscript 1 of the first root.

The general solution of the system has the form

$$x_{10}(t) = A_0 \cos\omega t + \frac{B_0}{\omega}\sin\omega t + E_0 + G_0 t$$

$$x_{20}(t) = p_1\left(A_0 \cos\omega t + \frac{B_0}{\omega}\sin\omega t\right) + p_2(E_0 + G_0 t)$$

where

$$p_1 = p_2^{(1)} = -\frac{c_{11} - \omega^2 a_{11}}{c_{12} - \omega^2 a_{12}} = -\frac{c_{21} - \omega^2 a_{21}}{c_{22} - \omega^2 a_{22}}, \quad p_2 = p_2^{(2)} = -\frac{c_{11}}{c_{12}} = -\frac{c_{21}}{c_{22}} \qquad (2.4)$$

The periodic solution of the generating system will not contain the term $G_0 t$. Furthermore, taking into account the fact that the system is autonomous, we may set $B_0 = 0$. In this manner we obtain

$$x_{10}(t) = A_0 \cos\omega t + E_0, \qquad x_{20}(t) = p_1 A_0 \cos\omega t + p_2 E_0 \tag{2.5}$$

For the original system (2.3) one may take the following initial conditions:

$$\begin{aligned} x_1(0) &= A_0 + \beta_1 + E_0 + \beta_2, \qquad x_1^{\cdot}(0) = \varphi(A_0 + \beta_1, E_0 + \beta_2, \mu) \\ x_2(0) &= p_1(A_0 + \beta_1) + p_2(E_0 + \beta_2), \quad x_2^{\cdot}(0) = p_2\varphi(A_0 + \beta_1, E_0 + \beta_2, \mu) \end{aligned} \tag{2.6}$$

Let us express the solution of system (2.3) in the form

$$x_1(t) = x^{(1)}(t) + x^{(2)}(t), \qquad x_2(t) = p_1 x^{(1)}(t) + p_2 x^{(2)}(t) \tag{2.7}$$

where

$$\begin{aligned} x^{(1)}(t) &= (A_0 + \beta_1)\cos\omega t + \sum_{n=1}^{\infty}\left[C_n^{(1)}(t) + \frac{\partial C_n^{(1)}}{\partial A_0}\beta_1 + \frac{\partial C_n^{(1)}}{\partial E_0}\beta_2 + \ldots\right]\mu^n \\ x^{(2)}(t) &= E_0 + \beta_2 + \varphi t + \sum_{n=1}^{\infty}\left[C_n^{(2)}(t) + \frac{\partial C_n^{(2)}}{\partial A_0}\beta_1 + \frac{\partial C_n^{(2)}}{\partial E_0}\beta_2 + \ldots\right]\mu^n \end{aligned} \tag{2.8}$$

The functions $C_n^{(1)}(t)$ and $C_n^{(2)}(t)$ are given by the formulas

$$C_n^{(1)}(t) = -\frac{1}{\Delta_0\omega^3}\int_0^t R_n^{(1)}(\tau)\sin\omega(t-\tau)\,d\tau, \qquad C_n^{(2)}(t) = \frac{1}{\Delta_0\omega^2}\int_0^t R_n^{(2)}(\tau)(t-\tau)\,d\tau \tag{2.9}$$

The formulas for the quantities $R_n^{(1)}(t)$ and $R_n^{(2)}(t)$ are given in [3].

We have the following periodicity conditions for the functions $x^{(1)}(t)$, $x^{(2)}(t)$ and their first derivatives:

$$\begin{aligned} x^{(1)}(T_0 + \alpha) &= A_0 + \beta_1, \qquad & x^{\cdot(1)}(T_0 + \alpha) &= 0 \\ x^{(2)}(T_0 + \alpha) &= E_0 + \beta_2, \qquad & x^{\cdot(2)}(T_0 + \alpha) &= \varphi \end{aligned} \tag{2.10}$$

Herein we denote by $T_0$ the period of the generating solution $T_0 = 2\pi/\omega$, and by $T_0 + \alpha$ the period of the solution of system (2.3). The quantity $\alpha$ can be represented in the form

$$\alpha = \sum_{n=1}^{\infty}\left[N_n(T_0) + \frac{\partial N_n}{\partial A_0}\beta_1 + \frac{\partial N_n}{\partial E_0}\beta_2 + \frac{1}{2}\frac{\partial^2 N_n}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n \tag{2.11}$$

The coefficients $N_n(T_0)$ are found with the aid of the condition $x^{\cdot(1)}(T_0 + \alpha) = 0$. We find that

$$N_1(T_0) = \frac{1}{\omega^2 A_0} C_1^{\cdot(1)}(T_0)$$
$$N_2(T_0) = \frac{1}{\omega^2 A_0} [C_2^{\cdot(1)}(T_0) + N_1 C_1^{\cdot\cdot(1)}(T_0)] \tag{2.12}$$
$$N_3(T_0) = \frac{1}{\omega^2 A_0} \left[C_3^{\cdot(1)}(T_0) + N_2 C_1^{\cdot\cdot(1)}(T_0) + N_1 C_2^{\cdot\cdot(1)}(T_0) + \frac{1}{2} N_1^2 C_1^{\cdot\cdot\cdot(1)}(T_0)\right]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Now we substitute the quantity $\alpha$ into the conditions: $x^{(1)}(T_0 + \alpha) = A_0 + \beta_1$ and $x^{\cdot(2)}(T_0 + \alpha) = \varphi$. These conditions take the form

$$\sum_{n=1}^{\infty} \left[ M_{jn}(T_0) + \frac{\partial M_{jn}}{\partial A_0} \beta_1 + \frac{\partial M_{jn}}{\partial E_0} \beta_2 + \frac{1}{2} \frac{\partial^2 M_{jn}}{\partial A_0^2} \beta_1^2 + \ldots \right] \mu^n = 0 \quad (j = 1, 2) \tag{2.13}$$

For the first one of these we obtain

$$M_{11}(T_0) = C_1^{(1)}(T_0)$$
$$M_{12}(T_0) = C_2^{(1)}(T_0) + \frac{1}{2} \omega^2 A_0 N_1^2 \tag{2.14}$$
$$M_{13}(T_0) = C_3^{(1)}(T_0) + \omega^2 A_0 N_1 N_2 - \frac{1}{2} N_1^2 C_1^{\cdot\cdot(1)}(T_0)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

For the second one of these conditions we find

$$M_{21}(T_0) = C_1^{\cdot(2)}(T_0)$$
$$M_{22}(T_0) = C_2^{\cdot(2)}(T_0) + N_1 C_1^{\cdot\cdot(2)}(T_0) \tag{2.15}$$
$$M_{23}(T) = C_3^{\cdot(2)}(T_0) + N_2 C_1^{\cdot\cdot(2)}(T_0) + N_1 C_2^{\cdot\cdot(2)}(T_0) + \frac{1}{2} N_1^2 C_1^{\cdot\cdot\cdot(2)}(T_0)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The system of equations

$$M_{11}(T_0) = C_1^{(1)}(T_0) = 0, \qquad M_{21}(T_0) = C_1^{\cdot(2)}(T_0) = 0 \tag{2.16}$$

determines the quantities $A_0$ and $E_0$. The form of the solutions of the system (2.3) depends on the multiplicity of the roots of the equations (2.16). If the Jacobian

$$\frac{D(M_{11}, M_{21})}{D(A_0, E_0)} \neq 0$$

then the parameters $\beta_1$ and $\beta_2$ can be expressed as power series in $\mu$

$$\beta_1 = \sum_{n=1}^{\infty} A_n \mu^n, \qquad \beta_2 = \sum_{n=1}^{\infty} E_n \mu^n \tag{2.17}$$

Hence the solution of system (2.3) will also be representable as a power series in $\mu$. If, however, the mentioned Jacobian is equal to zero then the analysis is performed as it was done for the nonautonomous system with one degree of freedom [4].

For the determination of the quantities $\varphi$, we express them in the form

$$\varphi = \sum_{n=1}^{\infty}\left[P_n(T_0) + \frac{\partial P_n}{\partial A_0}\beta_1 + \frac{\partial P_n}{\partial E_0}\beta_2 + \frac{1}{2}\frac{\partial^2 P_n}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n \qquad (2.18)$$

Let us substitute the quantities $\alpha$ and $\varphi$ into the remaining unused conditions of periodicity: $x^{(2)}(T_0 + \alpha) = E_0 + \beta_2$. Setting the coefficients of $\mu^n$ equal to zero, we obtain

$$\begin{gathered} T_0P_1(T_0) + C_1^{(2)}(T_0) = 0 \\ T_0P_2(T_0) + N_1P_1(T_0) + C_2^{(2)}(T_0) = 0 \\ T_0P_3(T_0) + N_1P_2(T_0) + N_2P_1(T_0) + C_3^{(2)}(T_0) - {}^1/_2 N_1^2 \ddot{C}_1^{(2)}(T_0) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \qquad (2.19)$$

From these equations one can determine the quantities $P_1(T_0)$, $P_2(T_0)$, ..., successively.

The remaining steps for completing the solution are obvious. In particular, the construction of a periodic solution with a period independent of $\mu$ is accomplished with the aid of the usual transformation of time.

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